PERFORMANCE OF A GALVANOTHERMOMAGNETIC COOLING ELEMENT UNDER STEADY-STATE THERMAL CONDITIONS

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The galvanothermomagnetic effect in a cylindrical cooling element is analyzed, with the nonuniform temperature distribution taken into account.

The galvanothermomagnetic (GTM) effect has been studied most thoroughly in single crystals of bismuth and in solid solutions of the bismuth–antimony system [1-9]. The prevailing theory of the GTM effect assumes a uniform temperature distribution over the cross section of the cooling element and is generally valid only for a gyrotropic medium. The material of a cooling element is strongly anisotropic [4, 6, 9, 10], however, so that assuming a uniform temperature distribution in the general case is not justified.

Generalized Equation of Heat Conduction. The law of energy conservation in the steady state for an anisotropic homogeneous medium with kinetic coefficients independent of the temperature can be written as [8]

$$-\varkappa_{ik}\frac{\partial^2 T}{\partial x_i \partial x_k}+\Delta\alpha_{ik}\frac{\partial T}{\partial x_k}J_i+\alpha_{ik}(-H)\,T\,\frac{\partial J}{\partial x_k}-\rho_{ik}J_iJ_k=0,$$

where x_i is the i-th coordinate (i, k = 1, 2, 3), $\Delta\alpha_{ik}=\alpha_{ik}(-H)-\alpha_{ik}(H)$, and $\alpha_{ik}(\pm H)$ is the thermal emf with the magnetic field in the forward direction (+H) and in the reverse direction (−H), respectively.

We consider the section of a long cylindrical cooling element made of a bismuth single crystal (Fig. 1). Let, as usually, the electric current flow in the direction of the trigonal axis, the magnetic field be oriented in the direction of the bisector axis, and the kinetic coefficient be independent of the temperature. The law of energy conservation can then be rewritten as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\varkappa_{22}}{\varkappa_{11}}\frac{\partial^2 T}{\partial y^2}-\frac{2Q_{31}JH}{\varkappa_{11}}\frac{\partial T}{\partial x}-\frac{\Delta\alpha_{32}J}{\varkappa_{11}}\frac{\partial T}{\partial y}+\frac{\rho_{33}J^2}{\varkappa_{11}}=0, \tag{1}$$

where Q_{31} is the Nernst coefficient and J is the density of the electric current in the z direction. In the general case Eq. (1) must also include a term proportional to the scalar derivative with respect to the temperature and representing the Righi–Leduc effect. In the case of bismuth this term can be disregarded.

The temperature distribution is generally also a function of the z coordinate. When the element is sufficiently long (about 2-3 cm), however, then this variation can be disregarded within the middle portion. This has been confirmed experimentally by means of a thermocouple recording the same temperature at the top surface of such an element along the z axis within the middle portion over distances to 2-3 cm from the center [1-3, 5, 7, 8]. We have verified this experimentally on specimens in the shape of circular cylinders [11].

We now rewrite Eq. (1) in the form

$$\frac{\partial^2 T}{\partial x^2}+K\frac{\partial^2 T}{\partial y^2}-L\frac{\partial T}{\partial x}+M=0, \tag{2}$$

where $\Delta\alpha_{32}=0$ [12], $K=\varkappa_{22}/\varkappa_{11}$, $L=2Q_{31}JH/\varkappa_{11}$, and $M=\rho_{33}J^2/\varkappa_{11}$. Equation (2) is the generalized equation of heat conduction, which describes the temperature distribution in our model. Inserting the expression

$$T(x, y)=V(x)+U(x, y)\exp\left(\frac{Lx}{2}\right), \tag{3}$$

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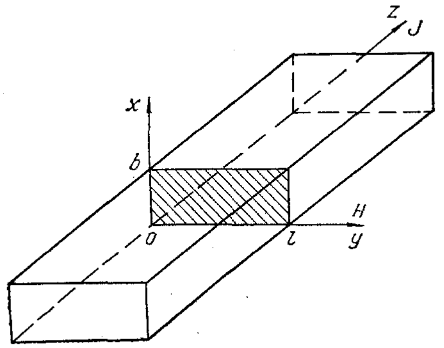

Fig. 1. Cylindrical cooling element made of a bismuth single crystal: x, binary axis; y, bissector axis; and z, trigonal axis; cross section is shaded.

into Eq. (2) yields

$$\frac{\partial^2 U}{\partial x^2} + K\frac{\partial^2 U}{\partial y^2} - \frac{1}{4}L^2 U = 0 \tag{4}$$

provided that function V(x) satisfies the equation

$$\frac{d^2 V}{dx^2} - L\frac{dV}{dx} + M = 0. \tag{5}$$

The general solution to Eqs. (4) and (5) can be expressed as, respectively,

$$U = \sum_{n=0}^{\infty}[A_n \exp(\beta_n y) + B_n \exp(-\beta_n y)] \sin\frac{2n+1}{2b}\pi x, \tag{6}$$

with $\beta_n = \frac{1}{2}\sqrt{\frac{L^2}{K} + \frac{(2n+1)^2}{Kb^2}\pi^2}$; for a cooling element with the x dimension equal to b, and

$$V(x) = A + \frac{M}{L}x + B\exp\left(\frac{Lx}{2}\right). \tag{7}$$

Constants A, B and A_n, B_n must be determined from the boundary conditions. Cooling usually proceeds with the element surfaces $y = 0$, $y = l$, $x = b$ adiabatically insulated from the ambient medium and heat exchange between the element base and the thermostat at temperature T_0 occurs isothermally. Mathematically, this means

$$T(0, y) = T_0, \tag{8}$$

$$q_x(b, y) = 0, \tag{9}$$

$$q_y(x, 0) = q_y(x, l) = 0. \tag{10}$$

Interesting is also the case where the adiabatic insulation of the lateral boundaries $y = 0$ and $y = l$ has been replaced by an isothermal contact with the thermostat. Unlike before, now heat exchange occurs between the lateral surfaces of the element and the thermostat at temperature T_0. This means the adiabatic boundary conditions (10) have been replaced by isothermal ones

$$T(x, 0) = T(x, l) = T_0. \tag{11}$$

In expressions (9) and (10) the quantities q_x and q_y are the components of the thermal flux density along the axes x and y, respectively. Explication of these components, with the aid of relation (3) and with the fact that $\alpha_{32}(-H) = \alpha_{32}(H) \neq 0$ taken into account, transforms the boundary conditions to

$$U(0, y) = 0, \tag{8'}$$

$$\frac{\partial U(b, y)}{\partial x} = 0, \tag{9'}$$

$$\frac{\partial U(x, 0)}{\partial y} - \frac{\alpha_{32}(H)J}{\varkappa_{22}}\left[V(x)\exp\left(-\frac{Lx}{2}\right) + U(x, 0)\right] =$$

$$= \frac{\partial U(x, l)}{\partial y} - \frac{\alpha_{32}(H)J}{\varkappa_{22}}\left[V(x)\exp\left(-\frac{Lx}{2}\right) + U(x, l)\right] = 0. \tag{10'}$$

In expressions (8')-(10') let

$$V(0) = T_0, \quad \frac{dV(b)}{dx} = \frac{L}{2} V(b),$$

making it possible to determine the constants A and B in expression (7):

$$A = T_0 - B, \quad B = \frac{L^2 T_0 + LMb - 2M}{L^2 [1 + \exp(Lb)]}.$$

With relation (3), the isothermal boundary conditions become

$$U(x, 0) = U(x, l) = [T_0 - V(x)] \exp\left(-\frac{Lx}{2}\right). \tag{11'}$$

The solutions for conditions (8) and (9) are satisfied automatically, inasmuch as

$$\sin \frac{2n+1}{2b} \pi x \Big|_{x=0} = \sin \frac{2n+1}{2b} \pi x \Big|_{x=b} = 0.$$

We will first examine the effect of adiabatic boundary conditions (10') on the GTM effect. We will use the Fourier series expansion

$$V(x) \exp\left(-\frac{Lx}{2}\right) = \sum_{n=0}^{\infty} C_n \sin \frac{2n+1}{2b} \pi x. \tag{12}$$

Such an expansion is legitimate [13]. The coefficients of this series are

$$C_n = \frac{2}{b} \int_0^b V(x) \exp\left(-\frac{Lx}{2}\right) \sin \frac{2n+1}{2b} \pi x dx.$$

Omitting here the simple intermediate calculations, we will write the final expressions for A_n, B_n, and C_n

$$A_n = \frac{1 - \exp(-\beta_n l)}{2 \operatorname{sh}(\beta_n l)} \frac{\alpha_{32}(H)}{\beta_n \varkappa_{22} - \alpha_{32}(H) J} C_n, \tag{13}$$

$$B_n = \frac{1 - \exp(\beta_n l)}{2 \operatorname{sh}(\beta_n l)} \frac{\alpha_{32}(H)}{\beta_n \varkappa_{22} - \alpha_{32}(H) J} C_n, \tag{14}$$

$$C_n = \frac{2A}{b} I_1 + \frac{2M}{Lb} I_2 + \frac{2B}{b} I_3, \tag{15}$$

where

$$I_1 = \frac{\frac{2n+1}{2b} \pi + (-1)^{n+1} \frac{L}{2} \exp\left(-\frac{Lb}{2}\right)}{\frac{L^2}{4} + \left(\frac{2n+1}{2b} \pi\right)^2};$$

$$I_2 = \frac{(-1)^n \frac{Lb}{2} \exp\left(\frac{Lb}{2}\right) - L\beta_n}{\frac{L^2}{4} + \left(\frac{2n+1}{2b} \pi\right)^2} + \frac{(-1)^n \left[\frac{L^2}{4} - \left(\frac{2n+1}{2b} \pi\right)^2\right]}{\left[\frac{L^2}{4} + \left(\frac{2n+1}{2b} \pi\right)^2\right]^2};$$

$$I_3 = \frac{\frac{2n+1}{2b} \pi + (-1)^{n+1} \frac{L}{2} \exp\left(\frac{Lb}{2}\right)}{\frac{L^2}{4} + \left(\frac{2n+1}{2b} \pi\right)^2}.$$

As can be seen, adiabatic insulation of the lateral surfaces results in a two-dimensional temperature distribution. This two dimensionality vanishes when $\alpha_{32}(H) = 0$, i.e., when the material of the cooling element is isotropic [14, 15].

In the case of isothermal boundary conditions, the heat transfer between element and thermostat occurs not only through the bottom base but also through the lateral surfaces so that the temperature distribution over the cross section becomes again two-dimensional. Constants A_n' and B_n' (the prime sign distinguishes them from constants (13) and (14) in the case of adiabatic boundary conditions) can be determined from conditions (11'):

$$A_n' = \frac{1 - \exp(-\beta_n l)}{2\,\mathrm{sh}(\beta_n l)}\, C_n', \tag{16}$$

$$B_n' = -\frac{1 - \exp(\beta_n l)}{2\,\mathrm{sh}(\beta_n l)}\, C_n'. \tag{17}$$

For finding A_n' and B_n' we have used the series expansion

$$[T_0 - V(x)] \exp\left(-\frac{Lx}{2}\right) = \sum_{n=0}^{\infty} C_n' \sin\frac{2n+1}{2b}\pi x,$$

where

$$C_n' = C_n + \frac{2T_0}{b} I_1,$$

with C_n and I_1 already determined earlier.

We now introduce the notation

$$U_n(y) = A_n \exp(\beta_n y) + B_n \exp(-\beta_n y).$$

Then the expression for the temperature distribution becomes

$$T(x, y) = V(x) + \exp\left(\frac{Lx}{2}\right) \sum_{n=0}^{\infty} U_n(y) \sin\frac{2n+1}{2b}\pi x. \tag{18}$$

We will analyze expression (18). In the case of adiabatic insulation of the lateral surfaces, we have for the middle portion of the element, i.e., for $y = l/2$ (at any x)

$$U_n(l/2) = \frac{2a^2}{J} \frac{C_n}{\mathrm{sh}(\beta_n l)} \frac{\mathrm{sh}(\beta_n l/2)}{\beta_n^2 - a^2}, \tag{19}$$

where $a = \alpha_{32}(H)J/\varkappa_{22}$. For bismuth $\varkappa_{11} \approx \varkappa_{22} = 0.1$ W/cm·K, $J \approx 10^2$ A/cm^2, and $\alpha_{32}(H) \approx 10^{-4}$ V/K so that $a \approx 0.1$ cm^{-1}. Also $L = 2Q_{31}HJ/\varkappa_{11} \approx 0.2$ cm^{-1} and $K \approx 1$. With these values one can write for β_n

$$\beta_n \approx \frac{2n+1}{2b}\pi.$$

The coefficients $U_n(l/2)$ corresponding to these parameters will be

$$U_n(l/2) = \frac{8a^2}{J} \frac{C_n b^2}{(2n+1)^2 \pi^2} \exp\left(-\frac{2n+1}{4b}\pi l\right), \tag{20}$$

assuming that $\beta_n l \approx (2n+1)\pi l/4b \gg 1$ or

$$b/l \ll (2n+1)\pi/4. \tag{21}$$

Condition (21) is known to be certainly satisfied for large n. For $n = 0$ it is satisfied quite closely when b is several millimeters large and $l = 1.5$–2 cm. The coefficient C_n in expression (20) is proportional to $1/(2n+1)$ and, therefore, decreases with increasing n. We thus conclude that coefficients $U_n(y)$ in expression (18) are, under the condition (21), small for the middle portion of the element. The series in expression (18) converges fast so that only two or three terms need to be retained for calculations. Owing to the smallness of its coefficients, this series does not, under condition (21), contribute appreciably to the temperature distribution. One can, therefore, regard the temperature distribution (18) as being described by function V(x) alone and thus to be one-dimensional. In summing up, it can be said that within the middle portion of the element, i.e., at points near the straight line $y = l/2$ the temperature distribution is one-dimensional when condition (21) is satisfied.

In the case of isothermal boundary conditions, with condition (21) also satisfied, we obtain for the middle portion of the element

$$U_n'(l/2) \approx 2C_n' \exp\left(-\frac{2n+1}{4b}\pi l\right),$$

where $C_n' \sim 1/(2n+1)$ under the same condition (21). We thus arrive at the general conclusion that, when condition (21) is satisfied, the temperature distribution within the middle portion of a cooling element is one-dimensional independently of the boundary conditions constraining its lateral surface.

Accordingly, the temperature distribution over a section of the cooling element is not one-dimensional. One-dimensionality is attainable within the middle portion of a cooling element with the proper combination of dimensions l and b or, in the case of a cooling element made of a gyrotropic material, with adiabatic boundary conditions imposed at its lateral surfaces.

Let us further assume that

$$U(x,\ l/2)\exp\left(\frac{Lx}{2}\right) \ll V(x).$$

Then

$$T(x) = T_0 + \frac{M}{L}\ x - \left[T_0 + \frac{Mb}{L} - \frac{2M}{L^2}\right] \frac{1-\exp(Lx)}{1-\exp(Lb)}.$$

Under the conditions of an experiment we have $Lb \ll 1$ and, therefore, at a point $x = b$

$$T(b) = T_0(2 + Lb)/2 + Mb^2/2.$$

Replacing L and M with their values, we obtain for the temperature drop

$$\Delta T = T_0 - T(b) = T_0 Q_{31} JHb/\varkappa_{11} - \rho_{33} J^2 b^2/2\varkappa_{11}.$$

It can be easily seen that the optimum current density

$$(\Delta T)_{\max} = ZT_0^2/2$$

corresponds to the maximum temperature drop

$$J_{\text{opt}} = Q_{31} HT_0/\rho_{33} b,$$

where $Z = Q_{31}^2 H^2/\varkappa_{11}\rho_{33}$ is the thermomagnetic figure of merit.

The expressions derived here do not differ formally from the already known ones [1-3, 5, 7, 8]. The fundamental difference is that they apply in the general case to a certain section: when the length and the width are large but the height is small. Physically this means that surfaces of a cooling element spaced sufficiently far from its center will not distort the linearity of the temperature distribution within its middle portion. In this sense, the earlier obtained results remain valid.

NOTATION

J, electric current density; H, magnetic field intensity; T, temperature; $\varkappa_{ik}$, α_{ik}, ρ_{ik}, components of thermal conductivity tensor, thermal emf tensor, and electrical resistivity tensor, respectively; x, y, z, Cartesian coordinates; $\varkappa_{11}$ and $\varkappa_{22}$, thermal conductivity along the binary axis and along the bisector axis, respectively; and b, l, dimensions of a cooling element.

LITERATURE CITED

1. B. J. O'Brien and C. S. Wallace, "Ettingshausen effect and thermomagnetic cooling," J. Appl. Phys., 29, 1010-1012 (1958).
2. C. F. Kooi et al., "Theory of the thermomagnetic cooler," Bull. Am. Phys. Soc., 8, 211-215 (1963).
3. C. F. Kooi et al., "Theory of the longitudinally isothermal Ettingshausen cooler," J. Appl. Phys., 34, 1735-1742 (1963).
4. K. F. Guff et al., "The thermomagnetic figure of merit and Ettingshausen cooling in bismuth—antimony alloys," Appl. Phys. Lett., 2, 145-186 (1963).
5. "New cooler uses thermomagnetic effect," Electronics, 36, 84-88 (1963).
6. H. J. Goldsmid, "Thermomagnetic energy conversion," GEC J., 31, 110-113 (1964).
7. G. L. Gurthie, "Optimization of the cross-sectional geometry of Nernst—Ettingshausen devices," J. Appl. Phys., 36, 3118-3122 (1965).
8. T. C. Harman and T. M. Honig, Thermoelectric and Thermomagnetic Effects and Their Applications, McGraw-Hill, New York (1967).

9. W. M. Yim and A. Amith, "Bismuth—antimony alloys for magnetothermoelectric cooling," Solid-State Electron., 15, 1141-1165 (1972).
10. H. J. Goldsmid, "Thermomagnetic energy conversion," Electron. Rev., 176, 514-518 (1965).
11. V. G. Okhrem and A. G. Samoilovich, "Physical principles of operation of a galvanothermomagnetic cooling element in the shape of a circular cylinder," Ukr. Fiz. Zh., 22, 39-44 (1977).
12. V. S. Vinnik and V. G. Okhrem, "Method of drift velocities and effect of inversion of the magnetic field on the thermal emf of bismuth," Fiz. Tverd. Tela, 18, 3411-3415 (1976).
13. G. M. Fikhtengolts, Study Course in Differential and Integral Calculus [in Russian], Vol. 3, Fizmatgiz, Moscow (1963).
14. Ya. I. Frenkel', Electrodynamics [in Russian], Vol. 2, ONTI, Moscow—Leningrad (1935).
15. A. G. Samoilovich and L. L. Korenblit, "Present status of the theory of thermoelectric and thermomagnetic effects in semiconductors," Usp. Fiz. Nauk, 49, 243-272 (1953).

OPTIMIZATION OF CURRENT LEAD WITH STRONG THERMAL INTERACTION WITH SURROUNDING STRUCTURAL ELEMENTS

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The article theoretically investigates the problem of optimizing a current lead according to the minimum heat flux at the cold end upon strong thermal interaction with the surrounding structural elements. It presents a number of generalized dependences characteristic of the optimum system.

In creating superconducting cryomagnetic systems for attaining high economic indicators, it becomes necessary to substantially reduce energy expenditure for compensating heat influxes into the cold zone. A number of authors [1-3] investigated in sufficient detail the problem of optimizing current leads without taking into account the effect of the structural elements surrounding them. However, it was shown in [4] that the temperature profiles along the current lead and the surrounding courses are fairly close to each other, which indicates considerable mutual thermal influence. The present article investigates the problems of optimizing current leads when there is considerable thermal interaction. As the initial assumption we use the assumption of ideal heat exchange, i.e., equal temperature profiles of the current lead, the cooling gas, and of the courses surrounding it which, according to [4], corresponds to a broad range of heat-transfer coefficients.

Let us examine the steady-state univariate equation of heat balance in the dimensionless form

$$\frac{d^2\Theta}{dx^2} = C\Theta_0' \frac{d\Theta}{dx} - \frac{1}{A} \cdot \frac{(\beta/\alpha)^2}{\beta/\alpha + 1}, \quad (1)$$

where

$$A = \frac{\Delta T}{\rho \frac{L}{S_\text{т}} \alpha} \cdot \frac{\beta}{\alpha}$$

is the dimensionless complex characterizing the thermal influence of the structural elements and is determined by the ratio of the amount of thermal energy transmitted through the structural elements by heat conduction to the amount of Joule heat released on the current lead whose thermal resistance is equal to the thermal resistance of the structural elements;

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